1 a) $M(t) \to 120 \Rightarrow P(t) \to M(t) \to 120$

b) $P'(0) = k(M(0) - P(0)) = 0.1(120 - 40) = 8$

$\hat{P}(t) = 40 + 8 * t$

c)(1) $M'(t) = 10e^{-0.1t} + 10te^{-0.1t} * (-0.1) = 0 \Rightarrow t_m = 10$

(2) $\hat{P}(t)$ at $t_m = 10$ $\quad P(10) \approx 120$

$P'(t) \quad P'(10) = 0.1(M(10) - P(10)) \approx 0.1(120 + 100e^{-1} - 120)$

$P'(10) > 0 \to$ still increasing, will reach max. later

d) $P_h'(t) = -kP_h(t) \Rightarrow P_h(t) = ce^{-0.1t}$

$P(t) = c(t)e^{-0.1t}$

$P'(t) = c'(t)e^{-0.1t} + \underline{c(t)e^{-0.1t} * (-0.1)}$

$= \underline{0.1}(120 + 10te^{-0.1t} \underline{- c(t)e^{-0.1t}})$

$\Rightarrow c'(t) = 12e^{0.1t} + t \Rightarrow c(t) = 120e^{0.1t} + \frac{1}{2}t^2 + d$

$\Rightarrow P(t) = (120e^{0.1t} + \frac{1}{2}t^2 + d)e^{-0.1t}$

$P(0) = (120 + d) * 1 = 40 \to d = -80$

$P(t) = (120e^{0.1t} + \frac{1}{2}t^2 - 80)e^{-0.1t}$

2a) $y_h'' + 0.1 y_h - 0.06 y_h = 0$

$\lambda^2 + 0.1\lambda - 0.06 = 0 \longrightarrow \lambda = \frac{2}{10}$ OR $\lambda = -\frac{3}{10}$

$y_h(x) = c_1 e^{2/10 x} + c_2 e^{-3/10 x}$

$y_p(x) = Ae^{-x} + B \qquad y_p'(x) = -Ae^{-x}$

$y_p''(x) = Ae^{-x}$

$\Rightarrow Ae^{-x} + 0.1(-Ae^{-x}) - 0.06(Ae^{-x} + B) = 21e^{-x} + 3$

$\Rightarrow 0.84A = 21 \rightarrow A = 25 \; ; \; -0.06B = 3 \rightarrow B = -50$

$y(x) = c_1 e^{2/10x} + c_2 e^{-3/10x} + 25e^{-x} - 50$

b) $y(0) = c_1 + c_2 + 25 - 50 = 0 \longrightarrow c_1 + c_2 = 25$

$y'(x) = \frac{2}{10} c_1 e^{2/10x} - \frac{3}{10} c_2 e^{-3/10x} - 25e^{-x}$

$c_1 = 75$
$c_2 = -50$

$y'(0) = \frac{2}{10} c_1 - \frac{3}{10} c_2 - 25 = 5 \longrightarrow \frac{2}{10} c_1 - \frac{3}{10} c_2 = 30$

$y(x) = 75e^{2/10x} - 50e^{-3/10x} + 25e^{-x} - 50$

3a) (1) $y(\frac{1}{2}) = y(0) + h\,2e^0 y(0) = 1 + \frac{1}{2}(2)(1)(1) = 2$

$y(1) = y(\frac{1}{2}) + h\,2e^{1/2} y(\frac{1}{2}) = 2 + \frac{1}{2}(2)(e^{1/2})(2) = 2 + 2e^{1/2}$

$y(\frac{3}{2}) = y(1) + h\,2e^1 y(1) = (2+2e^{1/2}) + \frac{1}{2}(2)(e)(2+2e^{1/2})$

$= 2 + 2e^{1/2} + 2e + 2e^{3/2}$

(2) if $y(0)=0 \rightarrow y(\frac{1}{2}) = y(1) = y(\frac{3}{2}) = \dots = 0$
numerical solution $y_n = 0$, matches analytic/exact solution

b) $y'(x) = 2e^x y(x)$ $\qquad y'(0) = 2(1)(1) = 2$

$y''(x) = 2e^x y(x) + 2e^x y'(x)$ $\qquad y''(0) = 2(1)(1) + 2(1)(2) = 6$

$y'''(x) = 2e^x y(x) + 2e^x y'(x) + 2e^x y'(x) + 2e^x y''(0)$ $\qquad y'''(0) = 2(1)(1) + 2(1)(2) + 2(1)(2) + 2(1)(6) = 22$

$y(x) = 1 + 2x + \frac{6}{2}x^2 + \frac{22}{6}x^3 = 1 + 2x + 3x^2 + \frac{11}{3}x^3$

<u>Alternative</u> : $y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ $\qquad y(0) = a_0 = 1$
$y'(x) = a_1 + 2a_2 x + 3a_3 x^2$

$$2e^x y(x) = 2(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots)(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots)$$
$$= 2a_0 + 2a_1 x + 2a_2 x^2 + 2a_3 x^3 + \dots$$
$$+ 2a_0 x + 2a_1 x^2 + 2a_2 x^3 + \dots$$
$$+ a_0 x^2 + a_1 x^3 + \dots$$

$$\Rightarrow \begin{cases} a_1 = 2a_0 = 2 \\ 2a_2 = 2a_1 + 2a_0 = 6 \\ 3a_3 = 2a_2 + 2a_1 + a_0 = 11 \end{cases} \Longrightarrow a_1 = 2,\ a_2 = 3,\ a_3 = \frac{11}{3}$$